Problem 5)

a) $$f(z)=\frac{1}{(x+iy)-(x_0+iy_0)}=\left[(x-x_0)+i(y-y_0)\right]^{-1}=\frac{(x-x_0)-i(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\Rightarrow$$

$$u(x,y)=\frac{(x-x_0)}{(x-x_0)^2+(y-y_0)^2}\quad,\quad v(x,y)=-\frac{(y-y_0)}{(x-x_0)^2+(y-y_0)^2}$$

$$\left\{\begin{array}{l}\dfrac{\partial u}{\partial x}=\dfrac{(x-x_0)^2+(y-y_0)^2-2(x-x_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\\[2ex] \dfrac{\partial v}{\partial y}=\dfrac{-(x-x_0)^2-(y-y_0)^2+2(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\end{array}\right.\Rightarrow\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ \checkmark$$

$$\left\{\begin{array}{l}\dfrac{\partial u}{\partial y}=\dfrac{-2(y-y_0)(x-x_0)}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\\[2ex] \dfrac{\partial v}{\partial x}=\dfrac{2(x-x_0)(y-y_0)}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\end{array}\right.\Rightarrow\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\ \checkmark$$

b) $$g(z)=e^{\alpha z}=e^{\overbrace{(\alpha'+i\alpha'')(x+iy)}}=e^{(\alpha'x-\alpha''y)+i(\alpha'y+\alpha''x)}$$

$$\left\{\begin{array}{l}u(x,y)=e^{(\alpha'x-\alpha''y)}\cos(\alpha'y+\alpha''x)\\[1ex] v(x,y)=e^{(\alpha'x-\alpha''y)}\sin(\alpha'y+\alpha''x)\end{array}\right.$$

$$\left\{\begin{array}{l}\dfrac{\partial u}{\partial x}=\alpha' e^{(\alpha'x-\alpha''y)}\cos(\alpha'y+\alpha''x)-\alpha'' e^{(\alpha'x-\alpha''y)}\sin(\alpha'y+\alpha''x)\\[2ex] \dfrac{\partial v}{\partial y}=-\alpha'' e^{(\alpha'x-\alpha''y)}\sin(\alpha'y+\alpha''x)+\alpha' e^{(\alpha'x-\alpha''y)}\cos(\alpha'y+\alpha''x)\end{array}\right.$$

$$\Rightarrow\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ \checkmark$$

$$\frac{\partial u}{\partial y}=-\alpha'' e^{(\alpha'x-\alpha''y)}\cos(\alpha'y+\alpha''x)-\alpha' e^{(\alpha'x-\alpha''y)}\sin(\alpha'y+\alpha''x)$$

$$\frac{\partial v}{\partial x} = \alpha' e^{(\alpha' x - \alpha'' y)} \sin(\alpha' y + \alpha'' x) + \alpha'' e^{(\alpha' x - \alpha'' y)} \cos(\alpha' y + \alpha'' x)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \checkmark$$

c) $h(z) = \ln(\beta z) = \ln(\beta) + \ln(z) = \ln(\beta) + \ln\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$

$$\Rightarrow \begin{cases} u(x,y) = \ln|\beta| + \frac{1}{2}\ln(x^2+y^2) \\ v(x,y) = \varphi_\beta + \tan^{-1}(y/x) \end{cases}$$

$$\eta = \tan^{-1}(\zeta) \Rightarrow \tan\eta = \zeta$$

$$\Rightarrow \frac{d}{d\zeta}(\tan\eta) = 1 \Rightarrow$$

$$\frac{d\eta}{d\zeta}(1+\tan^2\eta) = 1 \Rightarrow$$

$$\frac{d\eta}{d\zeta} = \frac{1}{1+\tan^2\eta} = \frac{1}{1+\zeta^2}$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \\ \frac{\partial v}{\partial y} = \frac{1}{x}\frac{1}{1+(y/x)^2} = \frac{x}{x^2+y^2} \end{cases} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark$$

$$\begin{cases} \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \\ \frac{\partial v}{\partial x} = -\frac{y}{x^2}\frac{1}{1+(y/x)^2} = -\frac{y}{x^2+y^2} \end{cases} \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \checkmark$$

In (a) we can find the derivative of $f(z)$ by setting $\Delta z = \Delta x$.

We'll have: $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{-(x-x_0)^2 + (y-y_0)^2 + 2i(x-x_0)(y-y_0)}{[(x-x_0)^2+(y-y_0)^2]^2}$

$$= -\frac{[(x-x_0) - i(y-y_0)]^2}{\{[(x-x_0)+i(y-y_0)][(x-x_0)-i(y-y_0)]\}^2} = -\frac{1}{[(x-x_0)+i(y-y_0)]^2} = -\frac{1}{(z-z_0)^2}$$

We would obtain the same result if we set $\Delta z = i\Delta y$, in which case $f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$. Either way, we see that the final result is the same as that obtained by applying the conventional rules of differentiation.

In (b) we find $g'(z)$ by setting $\Delta z = i\Delta y$. We'll have:

$$g'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = -\alpha'' e^{(\alpha' x - \alpha'' y)} \sin(\alpha' y + \alpha'' x) + \alpha' e^{(\alpha' x - \alpha'' y)} \cos(\alpha' y + \alpha'' x)$$

$$+ i\alpha'' e^{(\alpha' x - \alpha'' y)} \cos(\alpha' y + \alpha'' x) + i\alpha' e^{(\alpha' x - \alpha'' y)} \sin(\alpha' y + \alpha'' x)$$

$$= \alpha' e^{(\alpha' x - \alpha'' y)} [\cos(\alpha' y + \alpha'' x) + i \sin(\alpha' y + \alpha'' x)] + i\alpha'' e^{(\alpha' x - \alpha'' y)} \times$$

$$[\cos(\alpha' y + \alpha'' x) + i \sin(\alpha' y + \alpha'' x)] = (\alpha' + i\alpha'') e^{(\alpha' x - \alpha'' y)} e^{i(\alpha' y + \alpha'' x)}$$

$$= (\alpha' + i\alpha'') e^{\alpha'(x+iy) + i\alpha''(x+iy)} = (\alpha' + i\alpha'') e^{(\alpha' + i\alpha'')(x+iy)}$$

$$= \alpha e^{\alpha z}. \checkmark$$

Again, we would have obtained the same result had we started with $\Delta z = \Delta x$. Either way, the final result is the same as one would obtain using the conventional rules of differentiation.

In (c) we find $h'(z)$ by setting $\Delta z = \Delta x$. We'll have:

$$h'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x - iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z} \checkmark$$

This result could have been obtained using $\Delta z = i\Delta y$ as well. Also the conventional rules of differentiation yield:

$$\frac{d}{dz} \ln(\beta z) = \frac{\beta}{\beta z} = \frac{1}{z}. \checkmark$$